数理解析研究所講究録
679巻 1989年 261-274

A generalized Pohozaev identity and its applications

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§1. Introduction.

This is a joint work with Nichiro Kawano (Miyazaki Univ.) and Wei-Ming Ni (Univ. of Minnesota).

In this paper we establish a generalized Pohozaev identity and its variant for the following quasilinear elliptic equation,

$$(1.1)\qquad \operatorname{div}(A(|Du|)Du) + f(|x|,u) = 0$$

in $\mathbb{R}^n$, where Du is the gradient of u, $f(|x|,u)$ and $A(p)$ are given functions. The Pohozaev identity is useful to investigate the existence and non-existence of the ground state of (1.1). By a ground state we mean a positive solution u in $\mathbb{R}^n$, which tends to zero at ∞.

The Pohozaev identity was used by Pohozaev [15] in 1965 to show the non-existence of non-trivial solutions of non-linear eigenvalue problems for semi-linear elliptic equations. This kind of identities was first discovered by Rellich [17] in 1940 in his study of the first eigenvalue of Δ, and by Nehari [5] in 1963. The idea was applied to investigate the properties of solutions for non-linear elliptic equations (see, e.g., [1], [2], [3],

[4], [6], [7], [8], [9], [10], [11], [12], [13], [14], [16]). Especially, Ding and Ni [2] found that the Pohozaev-type identity is useful to get the non-existence theorems for the ground state in the anomalous case, $f_u(|x|,0) = 0$, by employing suitable change of variables. Recently, Ni and Serrin [9, 10, 11] established some generalized Pohozaev identies and used them to investigate the solutions of the quasilinear elliptic equations,

$$\mathrm{div}(A(|Du|)Du) + f(u) = 0. \tag{1.2}$$

They extend the argument employed by Ding and Ni to the quasilinear case. Their results are sharp, however their arguments are tricky and difficult. Our aim is to simplify, unify and generalize the method. We have found that the essence of the argument is clarified by rearranging the Pohozaev-type identity combined with the given equation (see, e.g. [18]).

§2. Main theorems.

We consider the radial solutions of (1.1). Let $u = u(r)$ be a radial solution of (1.1), then u satisfies the equation

$$r^{1-n}(r^{n-1}A(|u'|)u')' + f(r,u) = 0, \quad r > 0, \tag{2.1}$$

where n is a positive integer, and $u' = u'(r) = du(r)/dr$.

<u>Theorem 2.1</u>. <u>Suppose that</u> $A(p) \in C^1((0,\infty))$, $pA(p) \to 0$ <u>as</u> $p \to 0$, <u>and</u> $f(r,u)$, $f_r(r,u) \in C((0,\infty)\times(-\infty,\infty))$. <u>Let</u> $u(r) \in C([0,\infty)) \cap C^2((0,\infty))$ <u>satisfy</u> (2.1),

$$\text{(2.2)} \quad \lim_{r\to 0} r^n \int_0^{|u'(r)|} \rho E(\rho)\, d\rho = 0,$$

$$\text{(2.3)} \quad \lim_{r\to 0} r^{n-1} A(|u'(r)|)u'(r) = 0,$$

<u>and</u>

$$\text{(2.4)} \quad \lim_{r\to 0} r^n F(r,u(r)) = 0.$$

<u>Then the following generalized Pohozaev identity holds</u>:

$$\text{(PI)} \quad \begin{aligned} &R^n\{\int_0^{|u'(R)|} \rho E(\rho)\, d\rho + F(R,u(R)) + aR^{-1}A(|u'(R)|)u'(R)u(R)\} \\ &= \int_0^R \{n\int_0^{|u'|} \rho E(\rho)\, d\rho + (a + 1 - n)A(|u'|)|u'|^2 \\ &\qquad + nF(r,u) + rF_r(r,u) - auf(r,u)\}r^{n-1}dr \end{aligned}$$

<u>for all</u> $R > 0$, <u>where</u> a <u>is an arbitrary constant</u>,

$$F(r,u) = \int_0^u f(r,\xi)\, d\xi$$

<u>and</u>

$$E(p) = A(p) + p\,\frac{dA(p)}{dp}, \quad p > 0.$$

<u>Remark 2.1</u>. It hold that

$$\text{(2.5)} \quad \int_0^p \rho E(\rho)\, d\rho = \int_0^p \rho(\rho A(\rho))_\rho d\rho = Ap^2 - \int_0^p \rho A(\rho)\, d\rho,$$

since $pA(p) \to 0$ as $p \to 0$. Define

$$\varphi(p) = \begin{cases} pA(p), & p > 0, \\ 0, & p = 0, \end{cases}$$

and

$$\psi(p) = \begin{cases} p^2 A(p) - \int_0^p \rho A(\rho)\, d\rho, & p > 0, \\ 0, & p = 0, \end{cases}$$

For the sake of convenience, we simply denote $\varphi(p)$ and $\psi(p)$ by $pA(p)$ and $\int_0^p \rho E(\rho)\, d\rho$, respectively.

Remark 2.2. The equation (2.1) is equivalent to

(2.6) $\quad E(|u'|)u'' + \frac{n-1}{r} A(|u'|)u' + f(r,u) = 0$

for all $r > 0$ with $u' \neq 0$.

Remark 2.3. This identity is a variant of Theorem 2.1 in Ni and Serrin [9].

The following result is our main theorem.

Theorem 2.2. *Under the assumptions of Theorem 2.1, the following idenity holds*:

$$\sigma(\sigma+1)\{(m-1)A(|u'(R)|) - E(|u'(R)|)\}R^{n-2\sigma-2}w(R)^2$$

$$+ \{- R^2 w''(R)w(R) + 2\sigma R w'(R)w(R)\}\{E(|u'(R)|) - A(|u'(R)|)\}R^{n-2\sigma-2}$$

$$+ \{- R^2 w''(R)w(R) + (\tfrac{\lambda}{c}(\tfrac{n+k}{q+1} - 2(1-c)\sigma) - (n-1) + 2\sigma)Rw'(R)w(R)$$

$$+ \tfrac{\lambda}{c}(1-c)R^2 w'(R)^2\}R^{n-2\sigma-2}A(|u'(R)|)$$

$$+ \frac{\lambda}{c} R^n\{cA(|u'(R)|)|u'(R)|^2 - \int_0^{|u'(R)|} \rho A(\rho)\, d\rho\}$$

$$+ R^n\{\frac{\lambda}{c} F(R,u(R)) - u(R)f(R,u(R))\}$$

(PII)

$$= \frac{\lambda}{c}\int_0^R \{nF(r,u) + rF_r(r,u) - \frac{n+k}{q+1} uf(r,u) + \frac{(m+k)(1-\theta)}{m(q+1)}A(|u'|)|u'|^2$$

$$+ n(\frac{1}{m} A(|u'|)|u'|^2 - \int_0^{|u'|} \rho A(\rho)\, d\rho)\}r^{n-1}dr$$

<u>for all</u> $R > 0$ <u>with</u> $u'(R) \neq 0$, <u>where</u>

$$w(r) = r^{\sigma}u(r),$$

$$\sigma = \frac{m+k}{q+1-m}, \qquad q = \frac{(m-1)(n+k)(1-\theta)+\{(m-1)n+m+mk\}\theta}{n-m},$$

$$\lambda = \frac{c(q+1)\theta}{c(q+1)-(1-\theta)},$$

<u>and</u> m, k, θ, c <u>are arbitrary constants such that</u> $1 < m < n$, $k > -m$, $0 \leq \theta \leq 1$, $c > 0$, $cm \geq 1 - \theta$.

<u>Remark 2.4</u>. In fact, the above theorem holds for any choice of the constants m, k, θ, c as long as the other constants σ, q, λ are well-defined. However, applications to partial differential equations usually occur in the ranges restricted above.

<u>Remark 2.5</u>. We should note that

$$(m-1)A(p) - E(p) \to 0 \quad \text{as} \quad p \to 0$$

in the following important examples by choosing suitable m.

(i) The generalized Laplacian : $A(p) = p^{\mu-2}$.

Take $m = \mu$, then $(m-1)A - E = 0$.

(ii) The generalized mean curvature operator : $A(p) = (1+p^2)^{\mu/2-1}$.

Take $m = 2$, then $(m-1)A - E = (2-\mu)(1+p^2)^{\mu/2-2}p^2 \to 0$.

Therefore the coefficient of $R^{n-2\sigma-2}w(R)^2$ in (PII) vanishes as $u'(R) \to 0$. Actually we shall adjust λ so as to eliminate this coefficient (as $p \to 0$). The arrangement of the left-hand side of (PII) is closely related to Lemmas 4.1 and 4.2, which will appear later.

Remark 2.6. The case $\theta = 1$ and $c = 1/m$ is most important. In this situation $q = ((m-1)n+m+mk)/(n-m)$, $\sigma = (n-m)/m$ and $\lambda = 1$.

Remark 2.7. This theorem is very useful to obtain the non-existence theorems of the ground state in the anomalous case. The arguments employed in Ding and Ni [2, Theorem 5.13] and their generalizations to quasilinear equations in Ni and serrin [10, Theorems 4.1, 4.2, 5.3, 5.4, 6.5, 6.6] are considerably simplified by using the identity (PII). Furthermore we can naturally understand the meaning and sharpness of the assumptions in those papers. We shall see these facts in the subsequent sections.

§3. Applications to the generalized Laplace equation.

In this section we state some results obtained by virtue of generalized Pohozaev identity (PI) and (PII).

We shall treat the following generalized Laplace equation

$$(3.1) \quad \operatorname{div}(|Du|^{m-2}Du) + f(|x|,u) = 0, \qquad x \in \mathbb{R}^n,$$

where m is a constant. This equation corresponds to the case

(3.2) $A(p) = p^{m-2}$

in the equation (1.1). We are only interested in the positive radial solutions of (3.1). Thus we consider the ordinary differential equation

$$(F_\alpha) \quad \begin{cases} r^{1-n}(r^{n-1}|u'|^{m-2}u')' + f(r,u^+) = 0, & r > 0, \\ u(0) = \alpha > 0, \end{cases}$$

where $u^+ = \max\{u, 0\}$. We shall assume that

$$1 < m < n.$$

We now collect the hypotheses which will be assumed under various circumstances (but not simultaneously). Concerning the equation (F_α), we introduce

$$(F.1) \quad \begin{cases} f(r,u) \in C((0,\infty)\times[0,\infty)); \text{ and} \\ \text{for every } M, R > 0, \\ \sup\{r^{-\nu}|f(r,u)|: 0 < r \le R, 0 \le u \le M\} < \infty, \end{cases}$$

$$(F.2) \quad f(r,u) \ge 0 \quad \text{on} \quad (0,\infty)\times[0,\infty),$$

$$(F.3) \quad \begin{cases} \text{if } m > 2, \text{ then for every } L, M, R > 0, \\ \inf\{r^{-\nu}f(r,u): 0 < r \le R, L \le u \le M\} > 0, \end{cases}$$

$$(F.4) \quad \begin{cases} \text{for every } L, M, R > 0, \\ \sup\{r^{-\nu}|f_u(r,u)|: 0 < r \le R, L \le u \le M\} < \infty, \end{cases}$$

(F.5) $f_r(r,u) \in C((0,\infty)\times[0,\infty))$

(F.6) $nF(r,u) + rF_r(r,u) \le \frac{n-m}{m}\, uf(r,u)$ for all $u > 0$, $r > 0$,

(F.7) $f(r,u) \ge$ Pos. Const. $r^k u^q$ for all $u > 0$ and sufficiently large $r > 0$, where k and q are constants satisfying $k \ge -m$ and $q \ne m-1$,

(F.8) $nF(r,u) + rF_r(r,u) \ge \frac{n-m}{m}\, uf(r,u)$ for all $u > 0$, $r > 0$ with the strict inequality holds for some sequence of values u tending to zero and all sufficiently large $r > 0$,

(F.9) $uf(r,u) \ge mF(r,u)$ for all sufficiently small $u > 0$ and sufficiently large $r > 0$,

(F.2)* $\begin{cases} \text{there exists } \alpha_0 > 0 \text{ such that } f(r,u) < 0 \text{ (resp.} \\ f(r,u) > 0) \text{ for all } u > \alpha_0 \text{ (resp. } u < \alpha_0) \text{ and} \\ \text{sufficiently small } r > 0; \text{ and} \\ f(r,\alpha_0) = 0 \text{ for all sufficiently small } r > 0, \end{cases}$

(F.7)* $f(r,u) \ge$ Pos. Const. $r^k u^q$ for sufficiently small $u > 0$ and sufficiently large $r > 0$, where k and q are constants satisfying $k \ge -m$ and $q \ne m-1$,

where ν is a constant satisfying $\nu > -m$, and

$$F(r,u) = \int_0^u f(r,\xi^+)\, d\xi.$$

We state our results.

Theorem 3.1. _Suppose that_ (F.1)-(F.5) _hold_. _Then there exists a unique solution_ $u(r;\alpha) \in C([0,\infty)) \cap C^2((0,\infty))$ _of_ (F_α), _and_ $u(r) = u(r;\alpha)$ _satisfies the generalized Pohozaev-type identities_,

$$\begin{aligned} & R^n\{\frac{m-1}{m} |u'(R)|^m + F(R,u(R)) \\ (3.3) \qquad & \quad + \frac{n-m}{m} R^{-1}|u'(R)|^{m-2}u'(R)u(R)\} \\ & = \int_0^R \{nF(r,u(r)) + rF_r(r,u(r)) - \frac{n-m}{m} u(r)f(r,u^+(r))\}r^{n-1}\,dr \end{aligned}$$

and

$$\begin{aligned} & (m-2)\{-R^2w''(R)w(R) + 2\sigma Rw'(R)w(R)\}R^{n-2\sigma-2}|u'(R)|^{m-2} \\ & + \{-R^2w''(R)w(R) + (4\sigma-2m\sigma-m+1)Rw'(R)w(R) \\ & \quad + (m-1)R^2w'(R)^2\}R^{n-2\sigma-2}|u'(R)|^{m-2} \\ & + R^n\{mF(R,u(R)) - u(R)f(R,u^+(R))\} \\ (3.4) \qquad & \\ & = m\int_0^R \{nF(r,u(r)) + rF_r(r,u(r)) - \frac{n-m}{m} u(r)f(r,u^+(r))\}r^{n-1}\,dr \end{aligned}$$

where R _is an arbitrary positive number_, $w(r) = r^\sigma u(r)$ _and_ $\sigma = (n-m)/m$.

We shall investigate the properties of solutions of (F_α). The following theorem gives a sufficient condition for the existence of ground states.

<u>Theorem 3.2</u>. <u>Suppose that</u> (F.1)-(F.5) <u>and</u> (F.6) <u>hold</u>. <u>Then</u>, <u>for every</u> $\alpha > 0$, $u(r;\alpha)$ <u>is positive on</u> $[0,\infty)$.

<u>Moreover if</u> (F.7) <u>with</u> $q > m - 1$ <u>holds</u>, <u>then</u> $u(r;\alpha) \to 0$ <u>as</u> $r \to \infty$.

We shall also give some sufficient conditions for the solutions of the equation (F_α) having a zero.

<u>Theorem 3.3</u>. <u>Suppose that</u> (F.1)-(F.5), (F.7) <u>with</u> $q \le ((m-1)n+m+mk)/(n-m)$, (F.8) <u>and</u> (F.9) <u>hold</u>. <u>Then</u>, <u>for every</u> $\alpha > 0$, $u(r;\alpha)$ <u>has a finite zero on</u> $[0,\infty)$.

<u>Remark 3.1</u>. Theorems 3.2 and 3.3 are closely related to Theorems 3.2 and 4.1 in Ni and Serrin [10].

We now explain the meaning of the above theorems. Consider the equation,

$$(3.5) \qquad r^{1-n}(r^{n-1}|u'|^{m-2}u')' + r^k(u^+)^q = 0, \quad u(0) = \alpha > 0,$$

where $1 < m < n$, $k > -m$, and $q > m - 1$. For every $\alpha > 0$, (3.5) has a unique solution $u = u(r;\alpha) \in C([0,\infty)) \cap C^2((0,\infty))$ by Theorem 3.1. In view of Theorems 3.2 and 3.3, the structure of solutions are as follows;

(i) If $q \ge ((m-1)n+m+mk)/(n-m)$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and tends to zero as $r \to \infty$ for every $\alpha > 0$.

(ii) If $q < ((m-1)n+m+mk)/(n-m)$, then $u(r;\alpha)$ has a finite zero on $[0,\infty)$ for every $\alpha > 0$.

The case $f(r,u) = r^k u^{((m-1)n+m+mk)/(n-m)}$ lies on the borderline of the existence and non-existence. Here small perturbations can seriously affect the situation. If

$$f(r,u) = K(r)u^{((m-1)n+m+mk)/(n-m)},$$

where $K(r) = Q(r)r^k$, $Q \in C^1([0,\infty))$, $Q(r) > 0$ and $Q'(r) \le 0$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and tends to zero as $r \to \infty$ for every $\alpha > 0$. On the other hand, if

$$f(r,u) = \tilde{K}(r)u^{((m-1)n+m+mk)/(n-m)},$$

where $\tilde{K}(r) = \tilde{Q}(r)r^k$, $\tilde{Q} \in C^1([0,\infty))$, $\tilde{Q}(r) > 0$, $\tilde{Q}'(r) \ge 0$ and $\tilde{Q}'(r) \not\equiv 0$, then $u(r;\alpha)$ has a zero on $[0,\infty)$ for any $\alpha > 0$. These are obtained by applying Theorems 3.2 and 3.3.

We consider a different kind of perturbations to the nonlinearity $f(r,u) = r^k u^{((m-1)n+m+mk)/(n-m)}$. If

$$f(r,u) = r^k u^{((m-1)n+m+mk)/(n-m)} + \varepsilon r^k u^{q'},$$

where $\varepsilon > 0$, $k > -m$ and $q' > ((m-1)n+m+mk)/(n-m)$, then $u(r;\alpha)$ is positive on $[0,\infty)$ and tends to zero as $r \to \infty$ for every $\alpha > 0$ by Theorem 3.2 On the other hand, if

$$f(r,u) = r^k u^{((m-1)n+m+mk)/(n-m)} - \varepsilon r^k u^{q'},$$

where $\varepsilon > 0$, $k > -m$ and $q' > ((m-1)n+m+mk)/(n-m)$, then (F_α) has no ground state in the class $C([0,\infty)) \cap C^2((0,\infty))$ by the following result.

<u>Theorem 3.4</u>. <u>Suppose that</u> (F.1), $(F.2)^*$, (F.5), $(F.7)^*$ <u>with</u> $q \le ((m-1)n+m+mk)/(n-m)$, (F.8) <u>and</u> (F.9) <u>hold</u>. <u>Then</u> (F_α) <u>does not admit any positive solution in</u> $C([0,\infty)) \cap C^2((0,\infty))$ <u>which tends to zero as</u> $r \to \infty$.

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